Stat 344
Life Contingencies I

Chapter 6: Premiums

Premiums

Premiums are the amounts paid to the insurer in exchange for the benefits provided to the insured. The premium may consist of one or more payments.

- In traditional life insurance contracts, the premiums would often be a level series of payments made while the policy is in force.
- Some types of insurance policies (e.g., Universal Life) allow the policyholder a great deal of flexibility with respect to the timing and amounts of the policy payments.
- Payout annuities are often funded by a single premium.

Net premiums are premiums that, on average, exactly cover the benefit paid by the insurer.

Gross premiums are the premiums actually paid by the insured and are intended to cover expenses and profit, in addition to the benefit.

Future loss random variables

Before proceeding to calculating premiums, we first need to consider the present value of the insurer's future loss:

General Definition

The PV of an insurer's future loss is the PV of *future* outflows from the insurer minus the PV of the *future* inflows to the insurer.

Since the timing and/or amounts of these inflows and outflows will generally be contingent on the life status of the insured, the PV of the insurer's future loss is a *random variable*.

Future loss random variables

There are a couple specific future loss random variables that we will make use of when determining premiums:

$$L_0^n = (\text{PV of benefit outflows}) - (\text{PV of net premium inflows})$$

$$L_0^g = (\text{PV of benefit outflows} + \text{PV of expenses}) - (\text{PV of gross premium inflows})$$

Example: [50] buys a $300,000 whole life insurance policy, payable at the moment of death. The net premium is payable continuously until the moment of death at a rate of P per year. Then at the moment of policy issue (the subscript on L denotes the time at which we're calculating the future loss), we would have:

$$L_0^n = 300,000\, v^{T_{[50]}} - P\, \bar{a}_{\overline{T_{[50]}}|}$$

Approaches for calculating premiums

There are different approaches that can be used in order to determine premiums for a particular benefit. We'll discuss two particular ones here:

1. The **Equivalence Principle** states that the EPV of the "benefits" should be equal to the EPV of the "premiums"
   - This principle can be applied on either a net or gross basis — this will determine what is included in "benefits" and "premiums".

2. The **Portfolio Percentile Principle** sets the premium at a level that results in a fixed, specified probability of loss for the insurer, on a block of business, i.e., some large number of such policies.

Applying the equivalence principle — net case

If we wish to use the equivalence principle to determine net premiums, then we should set net premiums so that the EPV of the benefit amounts paid by the insurer is equal to the EPV of the net premiums paid to the insurer:

$$\text{EPV of benefit outflows} = \text{EPV of net premium inflows},$$

so that

$$E\left[L_0^n\right] = 0.$$

Suppose that (x) purchases a whole life insurance with \$1 benefit payable at the end of the year of death. Level premiums of amount P are payable annually at the beginning of the year for as long as (x) lives.

$L_0^n = 1v^{K_x+1} - P\ddot{a}_{\overline{K_x+1}|}$ $\qquad E(L_0^n) = A_x - P\ddot{a}_x$

1. Write down an expression for L_0^n in this case.
2. Find expressions for the mean and variance of L_0^n.
3. If P is determined using the Equivalence Principle, find P.

$$L_0^n = v^{K_x+1} - P\left(\frac{1-v^{K_x+1}}{d}\right) = v^{K_x+1}\left(1+\frac{P}{d}\right) - \frac{P}{d}$$

$$Var(L_0^n) = Var(v^{K_x+1})\left(1+\frac{P}{d}\right)^2 = \left({}^2A_x - A_x^2\right)\left(1+\frac{P}{d}\right)^2$$

$E(In) = P\ddot{a}_x$

$E(Out) = A_x$

$P\ddot{a}_x = A_x \qquad P = \dfrac{A_x}{\ddot{a}_x}$

Example

Write down an expression for L_0^n in this case.

Example

Find expressions for the mean and variance of L_0^n.

Example

If P is determined using the Equivalence Principle, find P.

Example

Suppose that (65) buys a \$500,000 20-year endowment insurance with death benefit payable at the end of the year of death and net premiums determined according to the equivalence principle.

1. If premiums are paid annually in advance, find the amount of the annual premium. [18,235.27]
2. Now suppose that the benefit is paid at the moment of death and the premiums are payable monthly. Find the amount of the monthly premium (under UDD). [18,992.17]

Example

If premiums are paid annually in advance, find the amount of the annual premium. [18,235.27]

$$E(In) = P\ddot{a}_{65:\overline{20|}}$$

$$E(Out) = 500000A_{65:\overline{20|}}$$

Now suppose that the benefit is paid at the moment of death and the premiums are payable monthly. Find the amount of the monthly premium (under UDD). [18,992.17]

$$E(In) = P\ddot{a}^{(12)}_{65:\overline{20}|}$$

$$E(Out) = 500000\bar{A}_{65:\overline{20}|}$$

$$\bar{A}_{65:\overline{20}|} \overset{UDD}{=} \frac{i}{\delta} A^{1}_{65:\overline{20}|} + {}_{20}E_{65}$$

$$\frac{i}{\delta}\left(A_{65:\overline{20}|} - {}_{20}E_{65}\right) + {}_{20}E_{65}$$

$$\ddot{a}^{(12)}_{65:\overline{20}|} = \alpha(12)\ddot{a}_{65:\overline{20}|} - \left(1 - {}_{20}E_{65}\right)\beta(12)$$

Example

(60) buys a policy that consists of a \$200,000 10-year term insurance (payable at the end of the year of death) combined with a \$300,000 20-year pure endowment. There are two premiums, at times 0 and 4, with the second premium being twice as much as the first. Find the amount of the first premium, using the equivalence principle. [37,048.01]

$E(In) = P + 2P v^4 {}_4p_{60}$ $E(Out) = 200000 A^{1}_{60:\overline{10}|} + 300000 \, {}_{20}E_{60}$

Using the equivalence principle, we can find the net premium for any arbitrary life-contingent benefits and any premium payment pattern.

Categories of expenses

Some of the typical expenses incurred by life insurance companies include (these are just some examples, not an all-inclusive list):

- Expenses incurred at or around the time of issue
  - underwriting
  - sales / commission
  - marketing
  - issue
- Ongoing expenses
  - overhead / home office
  - commission
  - premium tax
  - maintenance / billing
- Claims / termination expenses
  - claim payment expenses

Expense bases

Expenses are often expressed:

- per policy
- per unit (often $\$1,000$) of death benefit amount
- as a percent of premium

Depending on the situation, we may calculate expenses on one or more of these bases.

- For example, we may assume that expenses are \$100 per policy per year and 3% of gross premiums paid.

In practice, determining and allocating expenses is a time-intensive and non-trivial task, but for our purposes, we'll assume that all expenses are known and given.

Applying the equivalence principle — gross case

If we wish to use the equivalence principle to determine gross premiums, then we should set gross premiums so that the EPV of the benefit amounts paid by the insurer plus the EPV of expenses is equal to the EPV of the gross premiums paid to the insurer:

$$\begin{aligned}&\text{EPV of benefit outflows} + \text{EPV of expenses}\\&\qquad = \text{EPV of gross premium inflows},\end{aligned}$$

so that

$$E\left[L_0^g\right] = 0.$$

Suppose that (30) buys a $500,000 25-year term insurance with death benefit payable at the moment of death. Premiums are paid annually during the term period and determined according to the equivalence principle.

$$P = \frac{500000\bar{A}^{1}_{30:\overline{25|}} + 2500 + 50\ddot{a}_{30:\overline{25|}}}{0.85\ddot{a}_{30:\overline{25|}}}$$

Expenses:

- 15% of gross premiums
- $5 per $1,000 of death benefit (at time of issue)
- $50 maintenance expense (every year the policy is in force)

Find the annual gross premium P. [610.83]

$E(In) = P\ddot{a}_{30:\overline{25|}}$

$E(Out) = 500000\bar{A}^{1}_{30:\overline{25|}} + 0.15P\ddot{a}_{30:\overline{25|}} + 2500 + 50\ddot{a}_{30:\overline{25|}}$

Example

$L_0^g = 5000(12)\,\ddot{a}^{(12)}_{\overline{K_{55}^{(12)}+1/12}|} + 450 + 50\,\ddot{a}_{\overline{K_{55}+1}|} + 0.02P - P$

Suppose that (55) purchases a whole life annuity-due paying $5,000 per month with a single gross premium (P) paid at issue.

<u>Expenses:</u>

Maintenance costs (paid at the beginning of the policy year):

- $500 first year
- $50 second and later years

Commissions: 2% of premium

1. Write down an expression for L_0^g.
2. Using the equivalence principle, calculate P. [978,788.77]

$E(In) = P$ $\quad$ $E(Out) = 5000(12)\,\ddot{a}_{55}^{(12)} + 450 + 50\,\ddot{a}_{55} + 0.02P$

Example

Write down an expression for L_0^g.

Example

Using the equivalence principle, calculate P. [978,788.77]

Profits

Profits can be incorporated either implicitly or explicitly into the calculation of life insurance premiums.

Writing out expressions for our future loss random variables, we can see that their values depend on when the insured dies.

We can examine the future loss as a function of the future lifetime of the policyholder.

In general, for a life insurance policy, the insurer's loss will tend to decrease with the future lifetime of the insured. For annuities, the reverse will generally be true.

Future Loss RV — Life Insurance Example

Suppose that (40) buys a whole life insurance policy with \$100,000 death benefit payable at the end of the year of death. The annual premium P is payable at the beginning of each year that (40) is alive. Suppose we know[1] that $A_{40} = 0.12106$ and $\ddot{a}_{40} = 18.45774$.

Then we have

$$L_0^n = 100,000\, v^{K_{40}+1} - P\, \ddot{a}_{\overline{K_{40}+1|}} \tag{1}$$

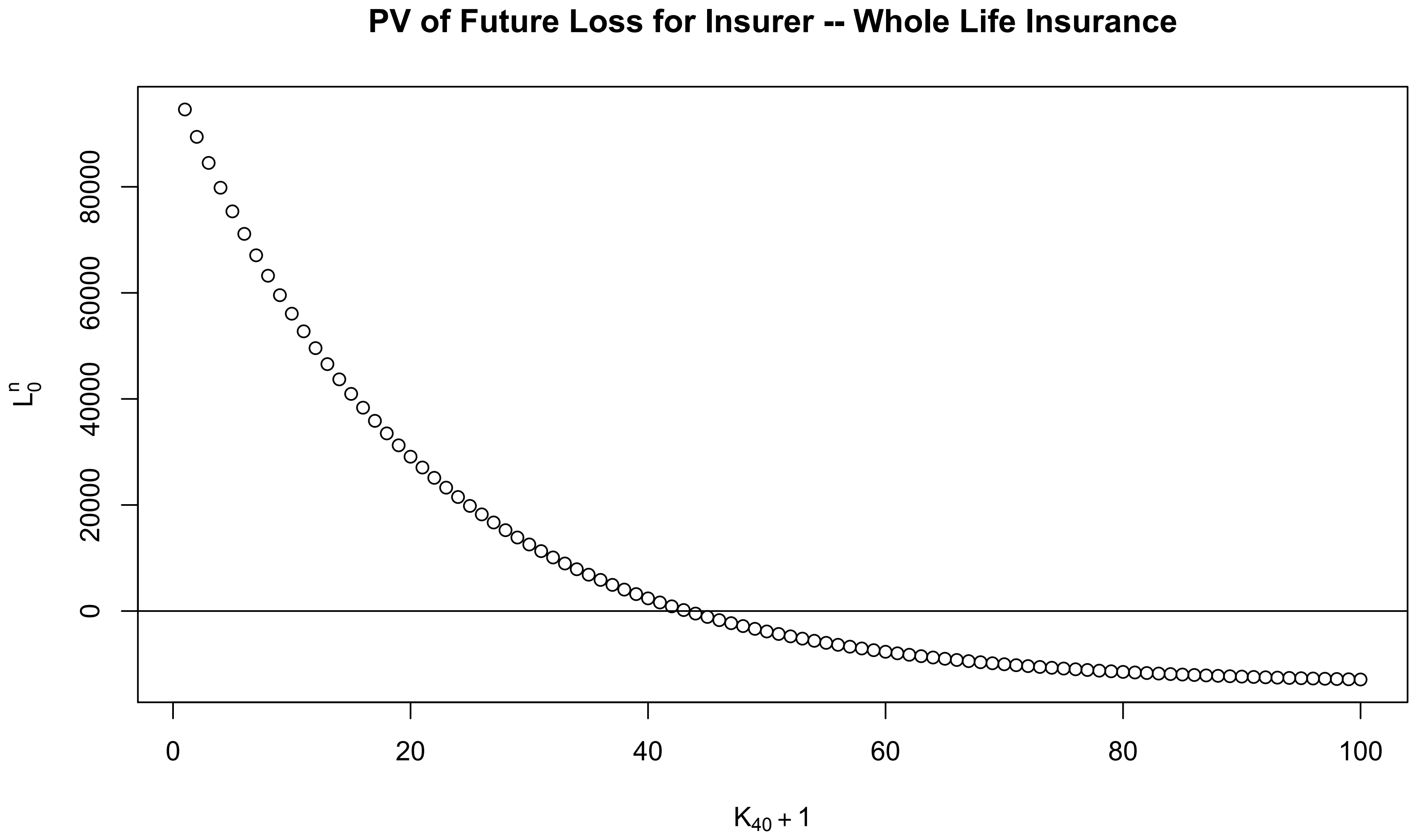
If P is calculated according to the Equivalence Principle, then

$$P = 100,000(0.12106)/18.45774 = 655.88.$$

[1] These were calculated under the assumption of the Standard Ultimate Mortality Model in Dickson et al. with $i = 5\%$.

Future Loss RV — Life Insurance Example (continued)

Looking at the value of L_0^n, as given in (1), as a function of the (curtate) future lifetime of (40) shows that the PV of the insurer's loss does indeed decrease as the insured lives longer:



Future Loss RV — Life Insurance Example (continued)

Since the premium P was determined by the Equivalence Principle, $E\left[L_0^n\right] = 0$, that is, the insurer will break even "on average". But what is the probability that the insurer loses money on this contract?

$$P\left[L_0^n > 0\right] = P\left[K_{40} \leq 42\right] = {}_{43}q_{40} = 0.319$$

From this, we can see that the distribution of L_0^n is skewed: there's a relatively small chance that the insurer will lose money (but when they do, it'll tend to be a large amount); and a large probability that the insurer will make money (but it will be a relatively small gain).

Future Loss RV — Annuity Example

Suppose that (65) buys a whole life annuity-due with annual payments of amount X, purchased by a single lump sum premium of $\$1,000,000$. Suppose we know[2] that $\ddot{a}_{65} = 13.5498$.

Then we have

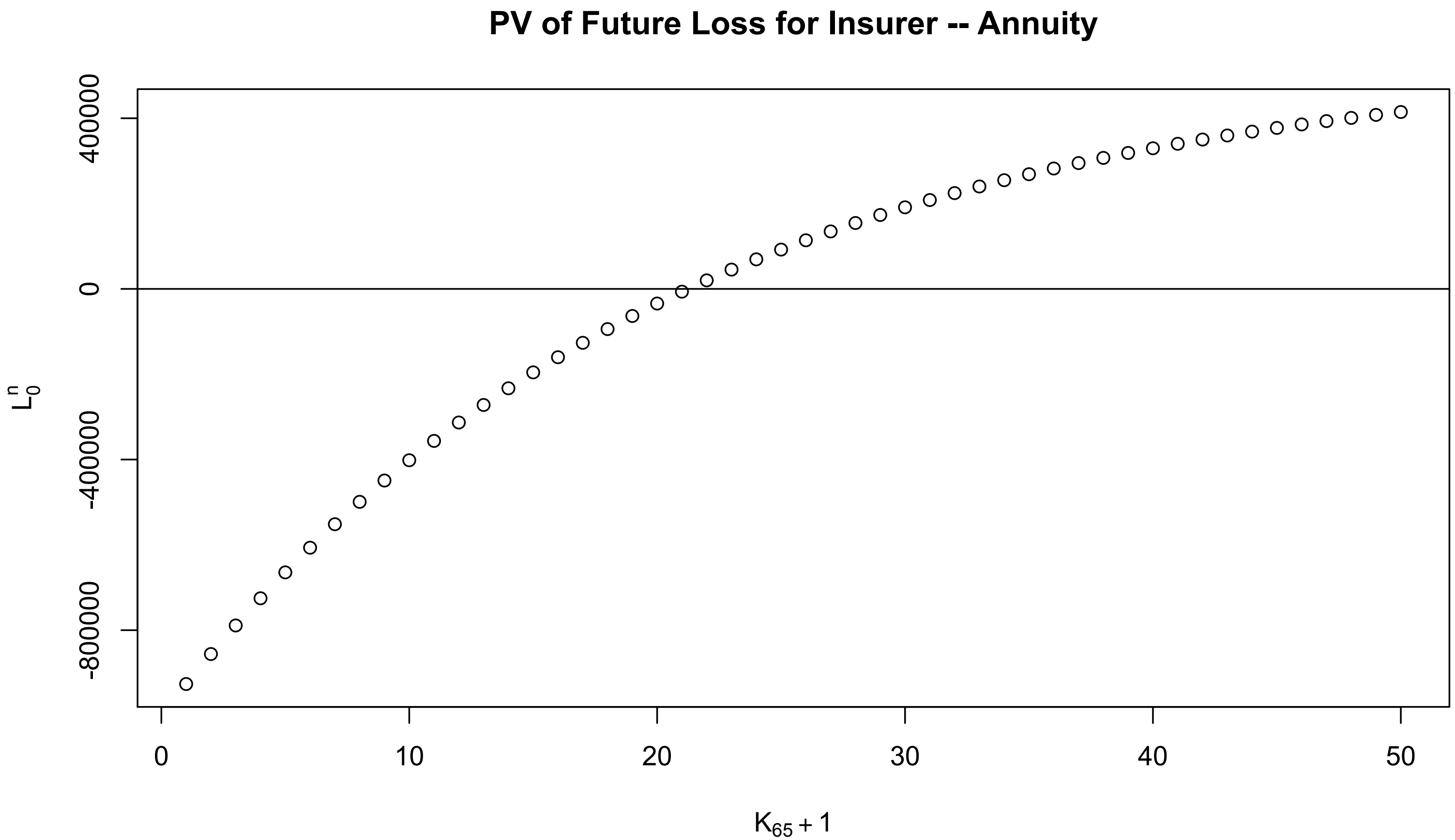
$$L_0^n = X\,\ddot{a}_{\overline{K_{65}+1}|} - 1,000,000 \tag{2}$$

If X is calculated according to the Equivalence Principle, then

$$X = 1,000,000/13.5498 = 73,801.66.$$

[2]As in the previous example, this was calculated under the assumption of the Standard Ultimate Mortality Model in Dickson et al. with $i = 5\%$.

Looking at the value of L_0^n, as given in (2), as a function of the (curtate) future lifetime of (65) shows that the PV of the insurer's loss does indeed increase as the insured lives longer, as opposed to the previous example:



Future Loss RV — Annuity Example (continued)

Since the premium P was determined by the Equivalence Principle, $E[L_0^n] = 0$, that is, the insurer will break even "on average". But what is the probability that the insurer loses money on this contract?

$$P[L_0^n > 0] = P[K_{65} \geq 21] = {}_{21}p_{65} = 0.6096$$

From this, we can see that the distribution of L_0^n is skewed, but in the opposite direction as in the previous example.

Portfolio Percentile Premium Principle

The Portfolio Percentile Premium Principle gives us a way to set premiums so that the probability of losing money on the entire block of business is set to a fixed value.

Assume we have N policies with independent and identically distributed (iid) future loss random variables $L_{0,1} \dots L_{0,N}$. Then we define the total future loss of the block as

$$L = \sum_{i=1}^{N} L_{0,i}.$$

Then $E[L] = N \cdot E[L_{0,1}]$ and $Var[L] = N \cdot Var[L_{0,1}]$. We want to set the premium for each policy so that $P[L < 0] = \alpha$.

- Then the probability of losing money is $1 - \alpha$.
- We typically set α to something large like 0.95.

Portfolio Percentile Premium Principle and the Central Limit Theorem

By the Central Limit Theorem, for large N, L has an approximately Normal distribution:

$$L \overset{\text{approx.}}{\sim} \mathcal{N}\left(N \cdot E[L_{0,1}], N \cdot Var[L_{0,1}]\right)$$

We can also scale and shift this into a "standard" Normal distribution:

$$\frac{L - (N \cdot E[L_{0,1}])}{\sqrt{N \cdot Var[L_{0,1}]}} \overset{\text{approx.}}{\sim} \mathcal{N}(0, 1)$$

This allows us to more easily calculate the desired probabilities.

Portfolio Percentile Premium Principle Example

Consider a whole life policy for a person age 60 with a death benefit of $1 payable at the end of the year of death. Premiums are paid at the beginning of each year, so long as the insured is alive.

We want to find the amount of the annual premium (P) using the Portfolio Percentile Premium Principle and assuming that $N = 100$ and $\alpha = 0.95$.

$$v^{K_{60}+1} - P\,\frac{1-v^{K_{60}+1}}{d} = v^{K_{60}+1} - \frac{P}{d} + \frac{P v^{K_{60}+1}}{d}$$

$$v^{K_{60}+1}\left(1+\frac{P}{d}\right) - \frac{P}{d}$$

Then $L_{0,i} = v^{K_{60}+1} - P\,\ddot{a}_{\overline{K_{60}+1}|}$, so that

$$\ddot{a}_{\overline{K_{60}+1}|} = \frac{1-v^{K_{60}+1}}{d}$$

$$E[L_{0,i}] = A_{60} - P\,\ddot{a}_{60} \quad \text{and} \quad Var(L_{0,i}) = \left(1+\frac{P}{d}\right)^2 \left[{}^2A_{60} - (A_{60})^2\right].$$

Portfolio Percentile Premium Principle Example (continued)

Using the Standard Ultimate Mortality Model with $i = 5\%$ gives the following values:

$$A_{60} = 0.29028 \qquad \ddot{a}_{60} = 14.90412 \qquad {}^2A_{60} = 0.10834$$

$$1.645\left(1+\frac{P}{0.0476}\right)\sqrt{2.4078} = -29.028 + 1490.41P \qquad 1.645 = -\frac{29.028 - 1490.41P}{\left(1+\frac{P}{0.0476}\right)\sqrt{(2.4078)}}$$

$$\frac{1.645P\sqrt{2.4}}{0.0476} - 1490.41P = 1.645\sqrt{2.4078} - 29.028$$

$$E[L] = 29.028 - 1,490.41\,P$$

$$Var(L) = \left(1 + \frac{P}{0.0476}\right)^2 (2.4078)$$

We want $P[L < 0] = 0.95$ which gives $1.645 = -\dfrac{E[L]}{\sqrt{Var[L]}}$.

Substituting the expressions above, we find $P = 0.02198$.

$P = 0.0194$

Accounting for Extra Risks in Pricing

If we want to set premium rates to reflect extra risks that may be present for a policyholder, we could adjust the underlying mortality assumption used. This can be done in a number of ways:

- Age Rating
- Additions to the force of mortality
- Scaling of the mortality (q_x) values

Then we can proceed to set premiums using any approach, using the adjusted mortality assumptions.

200,000 WL (65) SULT $i=0.05$

Exp

500 1st 100 renew

3% of all Gross premium

100 at termination

Calc net prem [5236.53]

gross prem [5534.72]

Net

$E(In) = P\ddot{a}_{65}$

$E(Out) = 200{,}000 A_{65}$

$L_0^n = 200{,}000 v^{K_{65}+1} - P\ddot{a}_{\overline{K_{65}+1|}}$

$E(L_0^n) = 200000 A_{65} - P\ddot{a}_{65}$

$Var(L_0^n) = \left(200000 + \frac{P}{d}\right)^2 \left({}^2A_{65} - A_{65}^2\right)$

Gross

$E(In) = P\ddot{a}_{65}$

$E(Out) = 200000 A_{65} + 100 A_{65} + 400 + 100\ddot{a}_{65} + 0.03P\ddot{a}_{65}$